Poincaré duality

§1. Rough idea: M^n manifold, $Y^k, Z^{n-k} \subset M$ transverse (all compact, oriented, smooth).

At each $p \in Y \cap Z$, we have $T_pY \oplus T_pZ \cong T_pM$.

Def. Intersection number

$$Y \cdot Z = \sum_{p \in Y \cap Z} \mathrm{sgn}(p), \quad \mathrm{sgn}(p) = \begin{cases} +1 & \text{if } T_pY \oplus T_pZ = T_pM \\ -1 & \text{if } \qquad\qquad = -T_pM. \end{cases}$$

This descends to a bilinear pairing on homology:

$$\begin{aligned} i : H_k(M) \otimes H_{n-k}(M) &\longrightarrow \mathbb{Z} \\ [Y] \otimes [Z] &\longmapsto Y \cdot Z. \end{aligned}$$

Note we'll always mean $H_k(M)/$torsion.

(Rmk Given $\alpha \in H_2(M^4)$, take $\mathbb{C} \to E \to M$ with $c(E) = \alpha$. A generic smooth section $s : M \to E$ will have $s^{-1}(0) \subset M$ a surface Σ with $[\Sigma] = e(E) = c(E) = \alpha$.)

Q: What can we say about this pairing?

Thm (Poincaré duality) i is nondegenerate: if $i(\alpha, \beta) = 0 \; \forall \beta$ then $\alpha = 0$.

Exercise if $n = 2k$ then $H_k(M) \otimes H_k(M) \longrightarrow \mathbb{Z}$

is $\begin{cases} \text{symmetric} & \text{if } k \text{ even} \\ \text{antisymmetric} & \text{if } k \text{ odd} \end{cases}$

The equivalence class $(H_k(M), i)$ is an invariant of M.

When $n = 4$:
- signature $\sigma(M) := \sigma(i) \in \mathbb{Z}$ is a bordism invariant.
- M smooth, i negative definite $\Rightarrow$ it's diagonalizable! (Donaldson)

  False if not smooth, e.g. E_8 manifold (Freedman).

§2. de Rham cohomology

A k-form on M is a section of $\Lambda^k T^*M$:

it's a multilinear, alternating form $\underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \to \mathbb{R} \quad \forall p \in M$.

The space $\Omega^k(M)$ of k-forms comes with

- wedge product $\wedge : \Omega^k(M) \otimes \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$
- exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

satisfying: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, $d^2 = 0$.

These can be defined in local coordinates: on $\mathbb{R}^n$ we have

$$\Omega^k(\mathbb{R}^n) = \operatorname{span}\{ f\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \mid f: \mathbb{R}^n \to \mathbb{R},\ 1 \le i_1 < i_2 < \dots < i_k \le n \}$$

with $(f dx_{i_1} \wedge \dots \wedge dx_{i_k}) \wedge (g dx_{j_1} \wedge \dots \wedge dx_{j_l}) = fg\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_l}$

and $$d(f dx_{i_1} \wedge \dots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$
$$= \left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \right) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

We extend integration from $\mathbb{R}^n$ to M: given a submanifold $X^k \subset M$,

$$\int_X : \Omega^k(M) \longrightarrow \mathbb{R}.$$

<u>Stokes's thm</u> if $\alpha \in \Omega^{k-1}(M)$ then $\int_X d\alpha = \int_{\partial X} \alpha$.

<u>Def.</u> $\alpha \in \Omega^k(M)$ is <u>closed</u> if $d\alpha = 0$

<u>exact</u> if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$.

The <u>de Rham cohomology</u> of M is

$$H^k_{dR}(M) = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}.$$

(By the Leibniz rule, $(H^*_{dR}(M), \wedge)$ is actually a graded ring!)

<u>Exercise</u> Use Stokes's theorem to prove that there is a well-defined bilinear pairing
$$H_k(M) \otimes H^k_{dR}(M) \longrightarrow \mathbb{R}$$
$$[X] \otimes [\omega] \longmapsto \int_X \omega.$$
Assume that every class in $H_k(M;\mathbb{Z})$ has a smooth representative.

<u>Ex.</u> A 0-form $f: M \to \mathbb{R}$ is closed $\Leftrightarrow df = 0 \Leftrightarrow f$ locally constant.

No exact 0-forms except 0, so $H^0_{dR}(M) \cong \mathbb{R}^{\#\text{components of } M}$.

<u>Ex.</u> If $n > \dim(M)$ then $\Omega^n(M) = 0$, so $H^n_{dR}(M) = 0$.

<u>Ex.</u> $\Omega^1(\mathbb{R}) = \{ f dx \}$. All such forms are <u>exact</u>, so $H^1_{dR}(\mathbb{R}) = 0$:

let $g(x) = \int_0^x f(t)\,dt \implies dg = g'(x)dx = f dx$.

Poincaré lemma $H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k>0 \end{cases}$ for all n.

Proof Induct on n. By assumption all closed k-forms $(k \geq 1)$ on $\mathbb{R}^n$ are exact, so look at $\omega \in \Omega^k(\mathbb{R}^{n+1})$.

We can write $\omega = \alpha_t + dt \wedge \beta_t \quad (t = x_{n+1})$

We construct $\eta_t \in \Omega^{k-1}(\mathbb{R}^n)$ with $\dot{\eta}_t = \beta_t$: $\eta_t = \int_0^t \beta_s ds$.

Then
$$\begin{aligned} d\eta_t &= d_{\mathbb{R}^n}\eta_t + dt \wedge \dot{\eta}_t \\ &= d_{\mathbb{R}^n}\eta_t + dt \wedge \beta \end{aligned}$$

so $\omega - d\eta_t = \alpha_t - d_{\mathbb{R}^n}\eta_t$.

If ω is closed then $d(\alpha_t - d_{\mathbb{R}^n}\eta_t) = 0$

$\Rightarrow \frac{d}{dt}(\alpha_t - d_{\mathbb{R}^n}\eta_t) = 0$ (no dt-term!)

$\Rightarrow \alpha_t - d_{\mathbb{R}^n}\eta_t$ defines a closed k-form on $\mathbb{R}^n$, independent of t.

By induction $\exists \xi \in \Omega^{k-1}(\mathbb{R}^n)$ with $d_{\mathbb{R}^n}\xi = \alpha_t - d_{\mathbb{R}^n}\eta_t$ and then $d(\eta_t + \xi) = \omega$, so ω is exact. ∎

Variant construction: $\Omega^k_c(M)$ = compactly supported k-forms on M.

$\rightsquigarrow$ compactly supported cohomology $H^k_c(M) := \ker(\partial)/\mathrm{im}(\partial)$.

Ex. $\ker(d: \Omega^0_c(\mathbb{R}^n) \to \Omega^1_c(\mathbb{R}^n)) = 0$, since constant + compact support $\Rightarrow 0$.

So $H^0_c(\mathbb{R}^n) = 0$.

[Exercise Prove that $\alpha \in \Omega^1_c(\mathbb{R})$ is exact iff $\int_{\mathbb{R}} \alpha = 0$.
Conclude that $H^1_c(\mathbb{R}) \cong \mathbb{R}$.

Poincaré lemma, part 2: $H^k_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k=n \\ 0 & \text{otherwise.} \end{cases}$

§3 Poincaré duality

Thm There is a perfect pairing $H^k_{dR}(M) \otimes H^{n-k}_c(M) \longrightarrow \mathbb{R}$

$$[\alpha] \otimes [\beta] \longmapsto \int_M \alpha \wedge \beta.$$

Equivalently: there is an isomorphism

$$PD: H^k_{dR}(M) \cong \left(H^{n-k}_c(M)\right)^*$$
$$[\alpha] \longmapsto \int_M \alpha \wedge - .$$

Pf sketch Let $\mathcal{U} = \{U_i\}$ be a good cover of M; induct on $|\mathcal{U}|$.

($|\mathcal{U}| = 1$ means $M = \mathbb{R}^n$: it's the Poincaré lemma.)

Write $U = U_1 \cup \dots \cup U_{m-1}$, $V = U_m$ and consider the Mayer–Vietoris sequence for $M = U \cup V$:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(U\cap V) \arrow[r] \arrow[d, "PD"] & H^k(M) \arrow[r] \arrow[d, "PD"] & H^k(U) \oplus H^k(V) \arrow[r] \arrow[d, "PD\oplus PD"] & H^k(U\cap V) \arrow[r] \arrow[d, "PD"] & \cdots \\
\cdots \arrow[r] & H^{n-k+1}_c(U\cap V)^* \arrow[r] & H^{n-k}_c(M)^* \arrow[r] & H^{n-k}_c(U)^* \oplus H^{n-k}_c(V)^* \arrow[r] & H^{n-k}_c(U\cap V)^* \arrow[r] & \cdots
\end{tikzcd}$$

and the PD maps for $U\cap V$, U, V are isomorphisms, so we apply the five lemma to conclude the same for M. ∎

There is a version of this for homology as well:

We compose $\int : H_k(M) \longrightarrow H^k_{dR}(M)^*$, $\quad [X] \longmapsto \int_X$

with the inverse of $PD: H^{n-k}_c(M) \xrightarrow{\cong} H^k_{dR}(M)^*$, $\quad [\beta] \longmapsto \int - \wedge \beta$

to get a linear map

$$\eta: H_k(M) \longrightarrow H^{n-k}_c(M)$$
$$[X] \longmapsto [\eta_X]$$

such that $\displaystyle\int_X \alpha = \int \alpha \wedge \eta_X$ for all $[\alpha] \in H^k_{dR}(X)$.

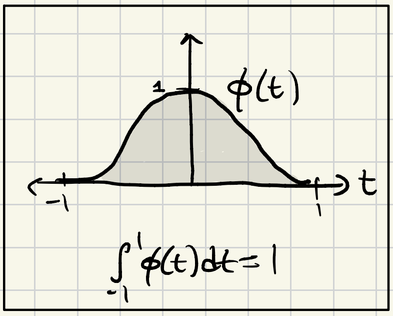
Thm (Poincaré duality, again)

The map $\eta : H_k(M) \longrightarrow H_c^{n-k}(M)$ is an isomorphism.

Q: can we construct the map η explicitly?

Ex. let $X^{n-1} \subset M^n$. Then X has a tubular neighborhood

$N(X) \cong X \times [-1,1]_t$



Define a 1-form η_X on M by

$$(\eta_X)_p(v) = \begin{cases} 0, & p \notin N(X) \\ \phi(t)\,dt, & p = (x,t) \in N(X). \end{cases}$$

Then $d\eta_X = 0$, so we get a class $[\eta_X] \in H_c^1(X)$.

Given $[\alpha] \in H^k(M)$, we now have

$$\int_X \alpha = \int_{X \times [-1,1]} \alpha \wedge \phi(t)\,dt = \int_{X \times [-1,1]} \alpha \wedge \eta_X = \int_M \alpha \wedge \eta_X,$$

so $[\eta_X]$ is the Poincaré dual of X.

Same idea works for any $X^k \subset M^n$ with trivial normal bundle,

or equivalently a product tubular neighborhood $N(X) \cong X \times \mathbb{R}^{n-k}$.

The general case is harder, but we still get η_X supported in $N(X)$.

On each normal fiber N_pX it looks like a bump function with volume 1.

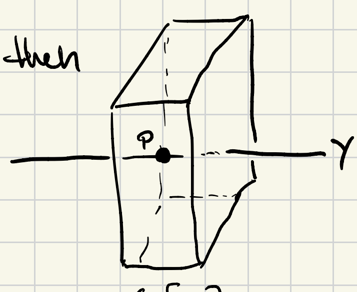
§4 Intersection forms

Exercise Show that if $Y^1 \subset M$ is transverse to X^{n-1}, then

$$\int_Y \eta_X = X \cdot Y.$$

Hint: work in explicit coordinates near

a single point $p \in X \cap Y$.



Suppose that $X^k, Y^{n-k} \subset M^n$ meet transversely.

We have closed forms $\eta_X \in \Omega_c^k(M)$, $\eta_Y \in \Omega_c^{n-k}(M)$

with $[\eta_X]$ and $[\eta_Y]$ Poincaré dual to X and Y.

The wedge product $\eta_X \wedge \eta_Y$ is only nonzero on

$N(X \cap Y)$: near any point $p \in X \cap Y$, we have local coordinates

$$X = \{x_{k+1} = x_{k+2} = \cdots = x_n = 0\}$$

$$Y = \{x_1 = x_2 = \cdots = x_k = 0\}$$

with $\eta_X = \phi_X(x_{k+1}, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_k$ $\quad \left(\int_{\mathbb{R}^{n-k}} \phi_X = 1\right)$

$\eta_Y = \phi_Y(x_1, \ldots, x_k)\, dx_{k+1} \wedge \cdots \wedge dx_n$. $\quad \left(\int_{\mathbb{R}^k} \phi_Y = 1\right)$

Then
$$\int_{N(p)} \eta_X \wedge \eta_Y = \begin{cases} 1 & \text{if } T_pX \oplus T_pY \cong +T_pM \\ -1 & \text{if } T_pX \oplus T_pY \cong -T_pM \end{cases}$$

is the sign of the intersection $X \cap Y$ at p.

Sum over p, and we have

Thm $\quad X \cdot Y = \int_M \eta_X \wedge \eta_Y$.

In other words, Poincaré duality identifies the wedge product on $H^*_{dR}(M)$ with the intersection product on $H_*(M)$.